MA3264 Mathematical Modelling

Thang Pang Ern

Contents

1. Models Using First-Order Differential Equations

1.1. *Modelling with Separable Differential Equations*

Mathematical models are rarely realistic. So, what is their purpose? The value lies in their ability to evolve. When a model predicts something nonsensical, it highlights where the simplifications fall short. You refine the model, adding complexity to better align it with reality. This iterative process mirrors life itself: start simple, and work towards something more realistic.

Take $\sin\theta = \theta$ as an example. Is it accurate? No, but it's a reasonable approximation for small θ. To improve it, we can use $\sin\theta = \theta - \theta^3/6$. Is that entirely true? Not quite, but it is closer to the truth. And with further refinements, we can get even better approximations.

Definition 1.1 (separable DE). A first-order differential equation is separable if it can be written as

$$M(x)\ dx = N(y)\ dy.$$

Notice that in this form, we say that we have *separated the variables* as everything involving x is on one side and everything involving y is on the other.

One should know how to solve such differential equations as shown in Definition 1.1 from one's high school days (or even MA2002) — simply integrate both sides, i.e.

$$\int M(x)\ dx = \int N(y)\ dy + c.$$

Example 1.1 (radioactive decay). Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with a sample containing 2 mg of this substance at certain time, say $t = 0$, what can be said about the amount available at a later time?

Solution. There could be a variety of radioactive materials present, and some of them might contribute to generating the substance we are analysing. As such, we shall deliberately disregard all other materials. So,

$$\frac{dm}{dt} = -km$$

where $m(t)$ is the amount of substance at time t, so $m(0) = 2$. Also, k is some arbitrary constant. With some algebraic manipulation, we have

$$\frac{1}{m}\,dm = -k\,dt.$$

Note that this admits the form in Definition 1.1, so one can integrate both sides to obtain

$$\ln\left(\frac{m}{c}\right) = -kt \quad \text{which implies} \quad m = ce^{-kt}.$$

Here, c is also an arbitrary constant. Setting $m(0) = 2$, we have $c = 2$, so $m = 2e^{-kt}$ — anyway, this means that the radioactive substance will decay at an exponential rate. This process is commonly known as an example of exponential decay. □

Example 1.2 (black holes). Stephen Hawking discovered that black holes lose mass over time, in addition to gaining it through the process of accreting matter. He developed a model to describe this complex phenomenon by simplifying the situation: he disregarded the details of matter falling into the black hole and concentrated solely on the radiation emitted, known as Hawking radiation. The rate of mass loss is described by the following differential equation:

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi G^2 M^2}$$

where t is time, M is the mass of the black hole, $\hbar$ is the reduced Planck's constant, c is the speed of light, and G is the universal gravitational constant.

One can easily compute the time T it takes for a black hole to disappear completely, i.e. the lifetime of a black hole with initial mass M_0. We have

$$T = \frac{5120\pi G^2 M_0^3}{\hbar c^4}.$$

Example 1.3 (planetary orbit). The orbit of a planet represents the path it follows as it moves around the Sun. In reality, this trajectory is highly complex due to gravitational influences from other planets, which pull on it from various directions. Isaac Newton, however, devised a simplified model of this situation by focusing exclusively on the interaction between the Sun and a single planet. He ignored the effects of other planets, asteroids, and miscellaneous items, as well as the fact that the Sun is not a perfect sphere, among other complexities. This approach allowed him to derive foundational insights into planetary motion.

In order to understand Newton's model of planetary orbits, one needs to recall polar coordinates (recall from MA2104)! Using his laws of motion, Newton discovered that a planet in his model has an orbit which satisfies the differential equation

$$\left(\frac{du}{d\theta}\right)^2 + (u-A)^2 = B^2,$$

where $u(\theta) = 1/r(\theta)$ and $A, B > 0$ are constants with $B/A < 1$. Note that $r(\theta)$ is the equation of our graph in polar coordinates.

This differential equation is separable, i.e. one can show that

$$d\theta = \frac{du/B}{\sqrt{1-\left(\frac{u-A}{B}\right)^2}}.$$

Integrating both sides yields

$$\theta + c = \arcsin\left(\frac{u-A}{B}\right).$$

Since $u = 1/r$, it follows that

$$r = \frac{1/A}{1+\frac{B}{A}\sin(\theta+c)}.$$

Since $B/A < 1$, we would see that this curve looks like an ellipse[†]! As such, in this simplified model of the solar system, all the planets have elliptical orbits (also known as Kepler's first law of planetary motion).

Example 1.4 (MA3264 AY24/25 Sem 1 Tutorial 1)**.** Solve the equation $y' = y, y(0) = 1$, in the following way: assume that y has an expansion of the form

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

and use the equation and the initial conditions to find the numbers a_n for all n. Next, consider the equation

$$y' = 2\sqrt{y} \quad \text{where} \quad y(x) \geq 0 \text{ and } y(0) = 0.$$

The previous method doesn't work. So find the solution in some other way.

Example 1.5 (MA3264 AY24/25 Sem 1 Tutorial 1)**.** One theory about the behaviour of moths states that they navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon. A certain moth flies near to a candle and mistakes it for the Moon. What will happen to the moth?

Hint: In polar coordinates (r, θ), the formula for the angle ψ between the radius vector and the velocity vector is given by

$$\tan \psi = r \frac{d\theta}{dr}.$$

If you wish to derive this formula, recall that the tangential component of a small displacement in polar coordinates $(r, \theta) \mapsto (r + dr, \theta + d\theta)$ is $r d\theta$ and the radial component is just dr. Use the formula to solve for r as a function of θ.

1.2. *Modelling with Linear Differential Equations*

Example 1.6 (melting ice)**.** The Arctic Ocean plays a crucial role in the global climate system, as it is the region most impacted by global warming. It is warming at approximately three times the rate of the rest of the planet, with the pace continuing to accelerate.

The surface of the Arctic Ocean consists of both ice-covered areas and open water. Let $I(t)$ represent the area covered by ice and $W(t)$ the area of open water, both as functions of time. The temperature $T(t)$ is also time-dependent. The rate of change of the ice-covered area, $I(t)$, is negatively influenced by the temperature $T(t)$, while the rate of change of the temperature is positively affected by $W(t)$. As such, we obtain the following pair of differential equations:

$$\frac{dI}{dt} = -aT \quad \text{and} \quad \frac{dT}{dt} = bW \quad \text{where } a, b > 0 \text{ are constants}$$

This relationship arises because ice, being highly reflective (a property known as having high albedo), reflects most sunlight, preventing it from absorbing significant heat. In contrast, open water, which

[†]Given the range of values of the ratio B/A, we can obtain various conic sections.

appears dark blue or nearly black, absorbs heat efficiently. Consequently, when $W(t)$ is large, more heat is absorbed, causing the temperature to rise.

The equations are called *linear* because of the absence of terms like T^3 or $\cos W$ — we only see T and W. The trick to solving such a pair of simultaneous equations is to differentiate one of the equations. In particular, we differentiate the first equation and substitute into the second one to obtain

$$\frac{d^2I}{dt^2} = -abW.$$

We note that the total area of the Arctic Ocean is a constant, which is equal to $I+W$. Differentiating a constant twice yields 0, so

$$\frac{d^2I}{dt^2} + \frac{d^2W}{dt^2} = 0 \quad \text{which implies} \quad \frac{d^2W}{dt^2} = abW.$$

We will learn how to solve such differential equations in Chapter 2. Anyway, one checks that

$$W(t) = Ae^{\sqrt{ab}t} + Be^{-\sqrt{ab}t} \quad \text{satisfies the differential equation.}$$

Here, A and B are some constants. Unless $A = 0$, the expression will blow up exponentially fast, with W increasing rapidly till it reaches the total area, and there will not be any ice at all. In fact it is feared that exactly this will happen some time this century!

Of course, this hasn't happened (yet), which suggests that there is a potential flaw in our model. However, nothing is actually wrong since it is just a model after all! The Arctic Ocean is an extraordinarily complex system governed by hundreds, if not thousands, of parameters and interrelated processes. Nevertheless, we need to begin somewhere.

First-order linear ODEs are very useful. However, the problem is that hey are not always separable. Having said that, there is a trick that allows us to solve them.

Proposition 1.1 (integrating factor). Consider linear differential equations of the form

$$\frac{dy}{dx} + yP(x) = Q(x),$$

where P and Q are functions of x. One can solve such differential equations by multiplying both sides of the equation by an integrating factor $\mu(x)$, then use the product rule, where

$$\mu(x) = \exp\left(\int P(t)\ dt\right).$$

Proposition 1.1 has already been covered in MA2002 so we will not discuss further. Now, what happens if the differential equation is neither separable nor linear? One nice instance is when we come across a Bernoulli equation (Definition 1.2).

Definition 1.2 (Bernoulli equation). The differential equation

$$\frac{dy}{dx} + yP(x) = Q(x)y^n,$$

where $n \in \mathbb{R}$, is a Bernoulli equation.

Again, Definition 1.2 has already been covered in MA2002 — the trick to solving such equations is to introduce the substitution $z = y^{1-n}$.

Example 1.7 (mixing problem). At time $t = 0$, a tank contains 2 kg of salt dissolved in 100 ℓ of water. Assuming that the water containing 0.25 kg of salt per litre is entering the tank at a rate of 3 ℓ/min and the well-stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t. Again, such questions have already been discussed in MA2002 so we will skip.

Example 1.8 (mixing problem). Imagine an experiment where a planet with a pristine atmosphere begins receiving 50 billion tons of CO_2 annually. The CO_2 mixes uniformly with the air, while biological and geological processes remove it, keeping the total atmospheric volume nearly constant. Based on what we have discussed thus far, the concentration of CO_2 would rise exponentially toward a limiting value.

Warned by their scientists, the planet's inhabitants immediately reduce the CO_2 concentration in their emissions at a rate inversely proportional to time.

Now, consider the following analogous problem. A tank contains 100 m^3 of pure air (negligible CO_2) at $t = 1$ second. At that moment, polluted air with a CO_2 concentration of $10/t\ \ell/\text{m}^3$ starts flowing in at 10 m^3/s. The mixture in the tank is pumped out at the same rate. Plot the quantity of CO_2 in the tank as a function of time.

Solution. We have

$$\frac{dQ}{dt} = \frac{100}{t} - \frac{Q}{10} \quad \text{with initial condition } Q(1) = 0.$$

The integrating factor is $e^{t/10}$ so we obtain

$$Q(t) = 100e^{-t/10}\left[\text{Ei}\left(\frac{t}{10}\right) - \text{Ei}\left(\frac{1}{10}\right)\right].$$

Here, $\text{Ei}(x)$ denotes the exponential integral (Definition 1.3).

Definition 1.3 (exponential integral). For real non-zero values of x, define $\text{Ei}(x)$ to be

$$\text{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt.$$

Figure 1 shows the graph of $Q(t)$.

We see that the amount of CO_2 in the tank increases for some time even though the concentration in

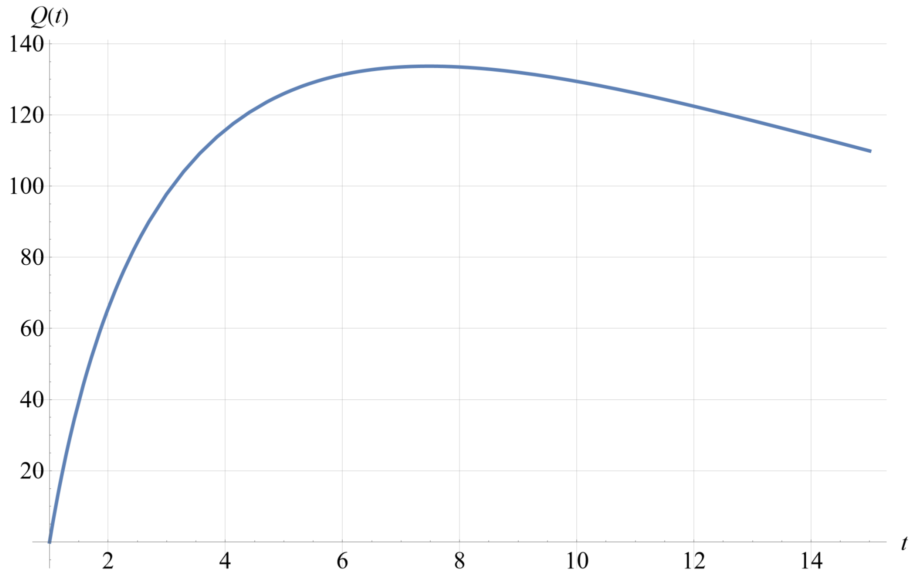

Figure 1: Graph of $Q(t)$

the gas entering the tank is decreasing. It reaches a rather high maximum, before decreasing rather slowly. This is known as the dreaded *momentum effect*, i.e. even if we start drastic reductions of CO_2 release now, the amount of it in the atmosphere will increase for a long time and will only be reduced to safe levels in the distant future[†]. □

Example 1.9 (radioactive decay). Sometimes, the product of radioactive decay is itself a radioactive substance that undergoes decay at a different rate. An example is uranium-thorium dating, a method used by paleontologists to estimate the age of fossils, particularly ancient corals.

Corals filter seawater, which contains trace amounts of uranium-234, a radioactive isotope. These corals absorb uranium-234 into their skeletons while alive. Over time, uranium-234 decays (to be precise, the type of radioactive decay that uranium-234 goes through is alpha decay) into thorium-230, another radioactive element via the equation

$$^{234}_{92}\text{U} \rightarrow ^{230}_{90}\text{Th} + ^{4}_{2}\alpha.$$

Uranium-234 has a half-life of 245,000 years, while thorium-230 has a shorter half-life of 75,000 years.

Thorium-230 is not naturally present in seawater, so when a coral dies, its skeleton contains uranium-234 but no thorium-230. This is because the coral's lifespan is negligible compared to uranium-234's half-life. However, over time, as uranium-234 decays, thorium-230 begins to accumulate in the coral skeleton. By measuring the ratio of uranium-234 to thorium-230 in a coral sample, we can estimate the time elapsed since the coral's death — its age.

[†]One can look up 'representative concentration pathway' for more notes on this

This information is crucial for understanding events like mass coral die-offs. If corals have historically died off regularly over long periods, it might suggest that current coral deaths are part of a natural cycle rather than solely caused by global warming.

To model this process, we make certain simplifying assumptions. Although other radioactive materials may be present, we ignore them because the decay products of thorium-230 typically decay much faster than uranium-234 or thorium-230 itself. Therefore, their contributions are negligible for our purposes.

Let $U(t)$ represent the number of uranium-234 atoms in the coral sample at time t, and $T(t)$ represent the number of thorium-230 atoms. Since each uranium-234 atom decay produces one thorium-230 atom, the rate at which thorium-230 is produced equals the rate at which uranium-234 decays. Consequently, we have the following relationships for the decay rates:

$$\frac{dU}{dt} = -k_U U \quad \text{and} \quad \frac{dT}{dt} = k_U U - k_T T,$$

where k_U and k_T are constants with $k_U \neq k_T$, and $U(0) = U_0$ and $T(0) = 0$. We wish to find t given that we know the ratio of $T(t)$ to $U(t)$ at the present time. Solving with the given data (first equation) yields

$$U = U_0 e^{-k_U t}.$$

One can attempt to solve for k_U and k_T, which are

$$k_U = \frac{\ln 2}{245,000} \quad \text{and} \quad k_T = \frac{\ln 2}{75,000}.$$

The second differential equation yields

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}.$$

Solving with $T(0) = 0$ yields

$$T(t) = \frac{k_U}{k_T - k_U} U_0 \left(e^{-k_U t} - e^{-k_T t} \right).$$

Although we do not know the value of U_0, we can consider the ratio T/U, which is

$$\frac{T}{U} = \frac{k_U}{k_T - k_U} \left[1 - e^{(k_U - k_T)t} \right].$$

So, if we compute the ratio T/U at the present time, we can solve for t and obtain our answer!

2. Models Using Second-Order Differential Equations

2.1. *Introduction*

We will need to study ordinary differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x),$$

where a,b,c are real constants and $f(x)$ is some given function. There is a systematic way of solving such ODEs.

Observe that since there are two derivatives present, we would need to integrate twice. Integrating once yields a constant, so the general solution of a second-order ODE must involve exactly two constants. To find them, we generally work with the initial conditions $y(0)$ and $y'(0)$ (which are usually known quantities). We then end up with two equations for two unknowns, which determine the two constants.

Definition 2.1 (characteristic equation). Consider the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Its characteristic equation is

$$a\lambda^2+b\lambda+c=0.$$

Note that in Definition 2.1, the characteristic equation is also known as the auxiliary equation. Note that here, we have taken $f(x)=0$. To find two solutions S_1 and S_2 to the equation, we consider the corresponding characteristic equation $a\lambda^2+b\lambda+c=0$. The idea here is that if λ is real, then $e^{\lambda x}$ is a solution. Usually, we obtain two solutions[†], i.e. two numbers λ_1 and λ_2. As such, we obtain two solutions $S_1=e^{\lambda_1 x}$ and $S_2=e^{\lambda_2 x}$.

However, two things can potentially go wrong.

- **Case 1:** The quadratic equation might have only one root λ (which must be real since a,b,c are real). Then, we will take $S_1=e^{\lambda x}$ and $S_2=xe^{\lambda x}$. One should verify by direct substitution that this indeed works.
- **Case 2:** We might obtain two solutions, which are complex. In fact, by the conjugate root theorem, the roots of the quadratic equation form conjugate pairs. We focus on one of them, and write it as

$$\lambda=\alpha+\beta i \quad \text{where } \alpha,\beta\in\mathbb{R}.$$

[†]Here is a fun exercise that is related to ST2131. Given the quadratic equation $Ax^2+Bx+C=0$ with $A,B,C\sim U(0,1)$, i.e. A,B,C are uniformly distributed on the interval $(0,1)$, what is the probability that the roots of the quadratic equation are real?

Then, we take

$$S_1 = e^{\alpha x}\cos\beta x \quad \text{and} \quad S_2 = e^{\alpha x}\sin\beta x.$$

Again, one should substitute these into the differential equation to be convinced that S_1 and S_2 are indeed solutions. Actually, these do not look so strange if we recall Euler's formula, which states that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

In all cases, we will be able to deduce the solutions S_1 and S_2, provided that a, b, c are constants.

For the case where $f(x) \neq 0$, suppose we have some *miraculous* way of deducing one solution to the differential equation. Call this solution $y = P(x)$, known as the particular solution. Then, we can find the general solution to the differential equation as follows. Consider

$$\begin{aligned} a\frac{d^2S_1}{dx^2} + b\frac{dS_1}{dx} + cS_1 &= 0 \\ a\frac{d^2S_2}{dx^2} + b\frac{dS_2}{dx} + cS_2 &= 0 \\ a\frac{d^2P}{dx^2} + b\frac{dP}{dx} + cP &= 0 \end{aligned}$$

We multiply the first equation by an arbitrary real number A and multiply the second by an arbitrary real number B. Thereafter, we add the three equations to obtain

$$a(AS_1 + BS_2 + P)'' + b(AS_1 + BS_2 + P)' + c(AS_1 + BS_2 + P) = f(x),$$

so we infer that $AS_1 + BS_2 + P$ is the general solution to the differential equation! This works due to the linearity of the derivative operator, and because the equation is also linear (inherently used the principle of superposition here). This method is not applicable to non-linear ODEs though.

Example 2.1. Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^{2x}.$$

Solution. Note that the differential equation can be written as

$$\frac{d^2y}{dx^2} - y = 0 + e^{2x}.$$

We first find the complementary solution. That is, the set of all y such that

$$\frac{d^2y}{dx^2} - y = 0.$$

The characteristic equation is $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$. Hence, $S_1 = e^x$ and $S_2 = e^{-x}$.

Now, we find the particular solution. The only way to obtain e^{2x} on the RHS is if it is already there on the LHS. As such, we try $P(x) = ce^{2x}$, where c has to be found. Since P satisfies the differential equation, we have

$$4ce^{2x} - ce^{2x} = e^{2x} \quad \text{which implies} \quad c = \frac{1}{3}.$$

Hence, $P = e^{2x}/3$. Combining the complementary solution and the particular solution yields the general solution, which is

$$y = Ae^x + Be^{-x} + \frac{1}{3}e^{2x}.$$

□

However, there are times where the method to finding the particular solution in Example 2.1 does not work. Let us take a look at Example 2.2.

Example 2.2. Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^x.$$

Finding the complementary solution is precisely the same as Example 2.1 since both differential equations share the same characteristic equation. Now, if we try $P(x) = ce^x$ as our particular solution, we will see that $0 = e^x$, which is an obvious error. As such, we need to amend the particular solution (the method to finding the particular solution is somewhat systematic) — try $P(x) = cxe^x$. We will see that

$$\frac{d^2P}{dx^2} - P = 2ce^x \quad \text{which implies} \quad 2c = 1.$$

Hence, $c = 1/2$ and the desired general solution is

$$y = Ae^x + Be^{-x} + \frac{1}{2}xe^x.$$

Examples 2.1 and 2.2 are great examples of finding particular solutions. The given method works because if we have an exponential function on the RHS, taking derivatives of exponential functions would give exponential functions. Similarly, it always works for polynomials and for products of exponential functions with polynomials. However, this method does not work if we have functions like $\tan x$ on the RHS!

Example 2.3. Now, what if we wish to solve a differential equation like

$$\frac{d^2y}{dx^2} + y = \cos x?$$

The easy way to handle this is to remember that $\cos x$ and $\sin x$ are really just names of the real and imaginary parts of e^{ix} respectively. As such, consider $z(x)$ to be a complex function such that $\text{Re}(z) = y$. Then, say we have the equation

$$\frac{d^2z}{dx^2} + z = e^{ix}.$$

This is easy to solve because we know what to do when we have an exponential function on the RHS! As such, we solve for z. As we are interested in y, upon finding z, we just take the real part of that.

Again, we first find the complementary solution. We first solve

$$\frac{d^2}{dx^2} + y = 0.$$

The characteristic equation is $\lambda^2 + 1 = 0$, so $\lambda = \pm i$. Hence,

$$S_1 = \cos x \quad \text{and} \quad S_2 = \sin x.$$

Next, try $P(x) = ce^{ix}$, which does not work. As such, we try $P(x) = cxe^{ix}$, for which we obtain $2ice^{ix} = e^{ix}$. So,

$$c = \frac{1}{2i} = -\frac{1}{2}i.$$

Hence,

$$P(x) = -\frac{1}{2}ixe^{ix} = -\frac{x}{2}(-\sin x + i\cos x).$$

The real part of P is $x\sin x/2$, so the general solution to the differential equation is

$$y = A\cos x + B\sin x + \frac{1}{2}x\sin x.$$

2.2. *Stability*

Definition 2.2 (pendulum equation)**.** Consider a pendulum. Let θ be the angle with the vertical and let L be the length of the pendulum. Then, using Physics (briefly see Figure 2), one can deduce that a differential equation governing θ is as follows:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

Sometimes, $d^2\theta/dt^2$ is written as $\ddot{\theta}$, which also denotes the second derivative of the angle with respect to time.

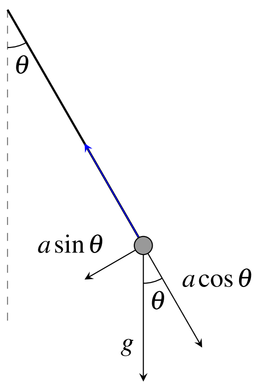

Figure 2: A free-body diagram of a pendulum bob

It is possible to solve the pendulum differential equation (Definition 2.2), but this involves complicated concepts in Mathematics like elliptic integrals.

Let us simplify the setup. An obvious solution is $\theta = 0$, which is known as an equilibrium solution, meaning that θ is a constant function. This means that if we set $\theta = 0$ initially, then θ will remain at 0 and the pendulum will not move — which of course know is correct. There is another equilibrium solution which is $\theta = \pi$. Again, in theory, if we set the pendulum exactly at $\theta = \pi$, then it will remain in that position forever. In reality, it will not due to gravity! As such, the equilibrium at $\theta = \pi$ is very much different from the one at $\theta = 0$ (an important distinction).

Definition 2.3 (equilibrium)**.** The equilibrium of an object is said to be stable if a small push away from equilibrium remains small. If the small push tends to grow large, then the equilibrium is unstable.

The concept of equilibrium is particularly important for engineers as they want vibrations of structures, engines, etc. to remain small.

We shall analyse the case where $\theta = \pi$. By Taylor's theorem, near $\theta = \pi$, we have

$$f(\theta) = f(\pi) + f'(\pi)(\theta - \pi) + \frac{1}{2} f''(\pi)(\theta - \pi)^2 + \ldots,$$

and upon letting $f(\theta) = \sin\theta$, we obtain the following series expansion:

$$\sin\theta = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 + \ldots$$

For small deviations away from π, note that $\theta - \pi$ is small, and $(\theta - \pi)^3$ is much smaller. So, we have the following approximation:

$$\sin\theta \approx -(\theta - \pi)$$

As such, the pendulum differential equation in Definition 2.2 can be approximated as follows:

$$\frac{d^2\theta}{dt^2} \approx \frac{g}{L}(\theta - \pi).$$

Using the substitution $\phi = \theta - \pi$, the differential equation can be written as

$$\frac{d^2\phi}{dt^2} = \frac{g}{L}\phi.$$

This equation has the general solution

$$\phi = Ae^{\sqrt{g/Lt}} + Be^{-\sqrt{g/Lt}}$$

so

$$\theta = Ae^{\sqrt{g/Lt}} + Be^{-\sqrt{g/Lt}} + \pi.$$

Since the exponential function grows very quickly, even if θ is close to π initially, it will not stay near it very long. Very soon, θ will arrive at either $\theta = 0$ or $\theta = 2\pi$, which is far away from $\theta = \pi$. This equilibrium is unstable! So, we ask how long would it take for things to get out of control? This is determined by the quantity in the exponent of the exponential term which is $\sqrt{g/L}$. Note that it takes longer for the pendulum to fall if L is large.

2.3. *Damped Oscillations*

When an object moves fairly slowly through air, the resistance due to friction is approximately proportional to its speed, and of course in the opposite direction. One would recall Hooke's law from H2 Physics. In fact, we can extend it to the following differential equation (Definition 2.4):

Definition 2.4 (simple harmonic oscillator).

$$m\frac{d^2x}{dt^2}+kx=0$$

This equation describes the oscillation of a block of mass m on one of a spring and a nail on the other end. Here, x measures how much the spring is stretched and k is a positive constant that measures the stiffness of the spring (known as the spring constant).

If we include friction which is proportional to the speed, we obtain

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0,$$

where b is a positive constant known as the damping coefficient. It quantifies the resistance to motion provided by the medium (such as air or fluid), often associated with dissipative forces like friction or drag. As such, the corresponding characteristic equation is

$$m\lambda^2+b\lambda+k=0.$$

Note that $m,b,k>0$. Let us discuss the solutions to this differential equation. We consider three cases on the nature of the roots.

- **Case 1:** λ_1 and λ_2 are real, which results in overdamping

  Example 2.4. Consider the differential equation

  $$\frac{d^2x}{dt^2}+3\frac{dx}{dt}+2x=0.$$

  Its characteristic equation is $\lambda^2+3\lambda+2=0$, which yields the roots $\lambda=-1$ and $\lambda=-2$. The general solution is

  $$x=B_1e^{-t}+B_2e^{-2t}.$$

  We see that the motion rapidly dies away to zero, which implies that there is much friction.

- **Case 2:** λ_1 and λ_2 are complex, which results in underdamping

  Example 2.5. Consider the differential equation

  $$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+13x=0.$$

  Its characteristic equation is $\lambda^2+4\lambda+13=0$, which yields the roots $\lambda=-2\pm 3i$. The general solution is

  $$x=B_1e^{-2t}\cos 3t+B_2e^{-2t}\sin 3t,$$

which by the R-formula (from O-Level Additional Mathematics), can be also written as

$$x = \sqrt{B_1^2 + B_2^2} e^{-2t} \cos\left(3t - \frac{\pi}{4}\right).$$

This acts like a simple harmonic oscillator, where the amplitude $\sqrt{B_1^2 + B_2^2} e^{-2t}$ is a function of time. Note that in this problem, there are two independent time scales. First, the factor e^{-2t} determines how quickly the oscillations decay over time. This decay is governed by the real part of the roots. Next, the angular frequency of oscillation is determined by the imaginary part of the roots. The oscillation period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3} \quad \text{where } \omega = 3 \text{ is the angular frequency.}$$

This represents the rapidity of oscillations within the decaying envelope.

2.4. *Forced Oscillations*

Now, consider the case where an external motor is attached to the block of mass m. This motor exerts a force of $F_0 \cos \alpha t$, where F_0 is the amplitude of the external force and α is the frequency. If $F_0 = 0$, then by Newton's second law, we have

$$m\frac{d^2x}{dt^2} + kx = 0$$

so we obtain the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}.$$

Here, ω is the frequency that the system has if we leave it alone, i.e. its natural frequency. It is not related to α.

If $F_0 \neq 0$, then we have

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \alpha t.$$

Let z be a complex function that satisfies the differential equation

$$m\frac{d^2z}{dt^2} + kz = F_0 e^{i\alpha t}.$$

The real part, $\operatorname{Re} z$, satisfies this differential equation, so we can solve for z and then take the real part. Try

$$z = ce^{i\alpha t} \quad \text{to be a solution.}$$

One can deduce that

$$c = \frac{F_0/m}{\omega^2 - \alpha^2} \quad \text{which implies} \quad \operatorname{Re} z = \frac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t.$$

We conclude that the general solution is

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos\alpha t,$$

where δ is some constant (we will explain in just a bit). Note that upon differentiation, we obtain

$$\frac{dx}{dt} = -A\omega\sin(\omega t - \delta) - \frac{aF_0/m}{\omega^2 - \alpha^2}\sin\alpha t.$$

The constants A and δ are fixed. One can deduce these values from $x(0)$ and $\dot{x}(0)$ as usual, where we recall that $\dot{x}(0)$ is dx/dt evaluated at $t = 0$.

Example 2.6. As an example, if $x(0) = \dot{x}(0) = 0$, then we have

$$A\cos\delta + \frac{F_0/m}{\omega^2 - \alpha^2} = 0 \quad \text{and} \quad A\omega\sin\delta = 0 \quad \text{respectively.}$$

Assuming that $F_0 \neq 0$, we cannot have $A = 0$, which forces $\delta = 0$. Hence,

$$A = -\frac{F_0/m}{\omega^2 - \alpha^2} \quad \text{which implies} \quad x = \frac{F_0/m}{\omega^2 - \alpha^2}(\cos\alpha t - \cos\omega t).$$

2.5. *The Phase Plane Method*

Newton's second law involves time derivatives, but sometimes it can be expressed in terms of spatial derivatives by using the chain rule. That is to say

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = \frac{d^2x}{dt^2}.$$

Recall Definition 2.4 on the differential equation governing simple harmonic oscillation, which is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

As such, we can rewrite it as follows:

$$m\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = -kx.$$

Integrating both sides yields

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = -\frac{1}{2}kx^2 + E,$$

where E is a constant. In fact this is not surprising as E is the total energy of the system! Since dx/dt denotes velocity, one would know that

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \text{ denotes the kinetic energy} \quad \text{and} \quad \frac{1}{2}kx^2$$

of the oscillator respectively. One should recall that the fact that E is constant is known as the conservation of energy.

This idea of turning time derivatives into space derivatives can be very useful when studying certain kinds of second-order non-linear differential equations. For example, we recall the pendulum problem (Definition 2.2) which is governed by the differential equation

$$\frac{d^2\theta}{dt^2}+\sin\theta=0 \quad \text{with initial conditions} \quad \theta(0)=1 \text{ and } \dot{\theta}(0)=1.$$

Here, we have taken $g/L=1$. Although we cannot find elementary solutions for this differential equation (recall that we can do so but the solution involving elliptic integrals would be non-elementary), we can still can gain some insights such as determining the maximum value of θ. This is simple as we have $\dot{\theta}(t)=0$. Solving yields $\theta_{\text{max}}\approx 1.53$.

In fact, there is a nice way of thinking about what we did. One can look at the equation involving $\dot{\theta}$ and use it to think of $\dot{\theta}$ as a function of θ. If we graph that function, we can see that the graph is a closed curve. As time goes by, the point $(\theta,\dot{\theta})$ moves around and around the closed curve. As such, the solution must be a periodic function of time. This makes sense as the physical system is a pendulum. We call this the phase plane method.

Example 2.7. We analyse the differential equation

$$\frac{d^2y}{dt^2}+\frac{1}{2}\cos y=0 \quad \text{with initial conditions} \quad y(0)=0 \text{ and } \dot{y}(0)=1.$$

Figure 3 shows the graphical solution to this differential equation. Note that the equation describes a non-linear oscillator, which should still typically produce bounded and oscillatory motion. In fact, for large t, $y(t)$ tends to infinity! What is really happening here?

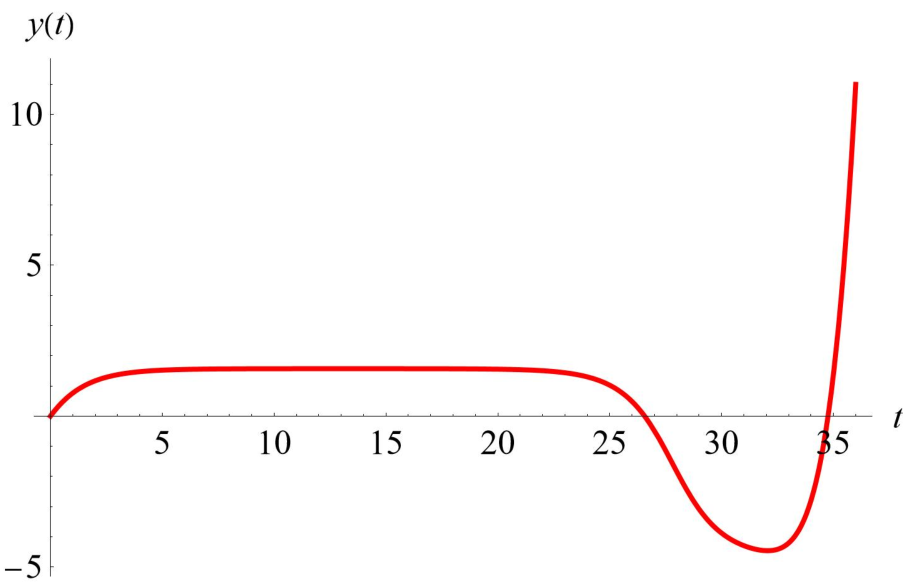

Figure 3: Solution to $\frac{d^2y}{dt^2}+\frac{1}{2}\cos y=0$ with initial conditions $y(0)=0$ and $y(0)=1$

We note that we can write the differential equation as

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2\right]+\frac{d}{dy}\left(\frac{1}{2}\sin y\right)\frac{dy}{dt}&=0\\ \frac{d}{dt}\left[\left(\frac{dy}{dt}\right)^2+\sin y\right]&=0\\ \left(\frac{dy}{dt}\right)^2+\sin y&=c\end{aligned}$$

Substituting the initial conditions yields $c=1$, so

$$\left(\frac{dy}{dt}\right)^2+\sin y=1,$$

which is the phase plane equation for the differential equation. On the $(y,\dot{y})$ phase plane, as the system moves from the point $(0,1)$ to the point $(\pi/2,0)$, it actually never gets there! One can use the method of separation of variables to obtain

$$t=\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin y}}\,dy=\int_0^{\pi/2}\frac{\sqrt{1+\sin y}}{\cos y}\,dy\geq\int_0^{\pi/2}\sec y\,dy$$

which is infinite. As such, the *correct* graph is not the one produced by Wolfram Mathematica (for instance), but rather the one in which y asymptotically approaches $\pi/2$. As such, the phase plane method helps us spot this error made by computers.

3. Population Models

3.1. *The Malthusian Model for Population Growth*

Population modelling is a crucial area of applied mathematics that uses differential equations to understand the dynamics of populations. These models can reveal surprising and sometimes counterintuitive behaviors in various species, from fish to humans.

The total population of a country, denoted as $N(t)$, is clearly a function of time. For simplicity, N can be measured in millions, meaning values less than 1 are still meaningful. Given the current population, can we predict how it will change? To begin, consider the per capita birth rate, B, which represents the number of babies born per second, divided by the total population at that moment. The value of B varies — it might be small in a populous country or large in a smaller one, depending on societal factors like cultural attitudes toward marriage and children. B could depend on time t and the current population N.

For simplicity, we assume that B is constant, i.e. people will always have as many children as possible, regardless of time or population size. In this case, the number of births over a small time interval δt is given by $BN\delta t$.

Similarly, consider the per capita death rate D, which also depends on t (i.e. better healthcare) or N (i.e. overcrowding). Assuming D is constant, the number of deaths over δt is $DN\delta t$.

Assuming no immigration or emigration, the change in population, δN, over δt is simply the difference between births and deaths. That is,

$$\delta N = \text{births} - \text{deaths} = (B-D)N\delta t.$$

Recall from MA2002 that we can divide throughout by δt and take the limit as $\delta t \to 0$. We then obtain the differential equation

$$\frac{dN}{dt} = (B-D)N = kN \quad \text{where } k \text{ is the net growth rate.}$$

This simple model was first proposed by Thomas Malthus in 1798, laying the foundation for what is now known as Malthusian population growth (Definition 3.1).

Definition 3.1 (Malthusian growth model)**.** Let N denote the current population, B and D denote the birth rate and death rate respectively. Then,

$$\frac{dN}{dt} = (B-D)N.$$

The Malthusian model predicts exponential growth if $k > 0$ or exponential decay if $k < 0$, assuming constant birth and death rates. To see why, one can easily solve the differential equation in Definition 3.1 to deduce that

$$N = N_0 e^{kt} \quad \text{where } k = B - D.$$

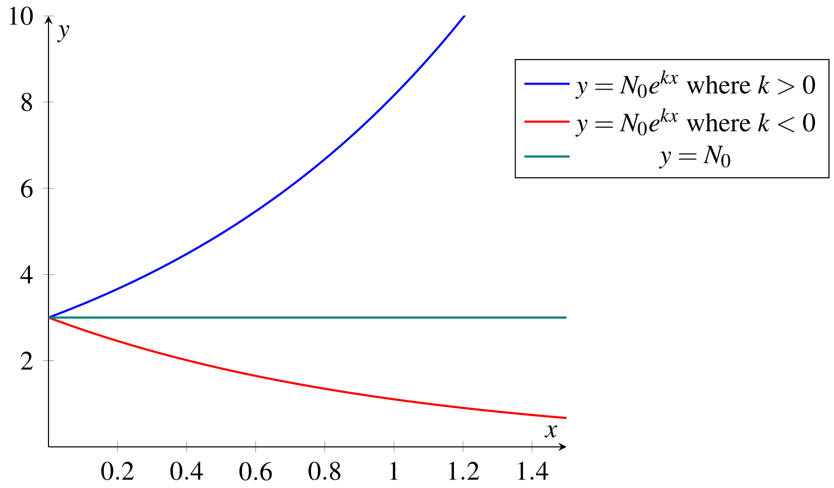

Figure 4: Interpretation of the Malthusian growth model

Malthus' model is interesting as it shows that static behaviour patterns can lead to disaster. As e^{kt} grows so quickly for $k > 0$, Malthus' assumptions must eventually go wrong — obviously there is a limit to the possible population. Eventually, if we do not control B, D will have to increase, so we have to assume that D is a function of N. Hence, we turn to Verhulst's model, which will be discussed in the next section.

3.2. *Verhulst's Model of Population Growth*

Previously, we mentioned that the death rate D, should depend on N. A natural starting point is the simplest possible choice, i.e.

$$D = sN \quad \text{where } s \text{ is a constant.}$$

This assumption is often referred to as the logistic assumption. It captures the idea that finite resources in the environment lead to higher death rates as the population increases due to factors like starvation and disease. Hence, we obtain Verhulst's logistic growth model, which was proposed by Pierre-François Verhulst in the 19th century.

Definition 3.2 (Verhulst's logistic growth model)**.** Again, let N denote the current population, B and D denote the birth rate and death rate respectively. Then, we can write

$$\frac{dN}{dt} = BN - DN = BN - sN^2 = BN\left(1 - \frac{sN}{B}\right).$$

We shall analyse Verhulst's growth model. Suppose the initial population N_0 is small. Then, $N(t)$ will remain small as well. Since N^2 becomes negligible compared to N, the equation simplifies to

$$\frac{dN}{dt} \approx BN \quad \text{which has solution} \quad N = N_0 e^{Bt}.$$

Thus, for small populations, the growth is approximately exponential, as predicted by Malthus.

As the population grows, the quadratic term sN^2 dominates as N^2 increases much faster than N. At some point, the terms BN and sN^2 balance, i.e. $BN = sN^2$. This happens when

$$N \approx \frac{B}{s}.$$

At this population size, the growth rate dN/dt becomes zero, indicating that the population stabilises. As such, the quantity B/s would be of interest.

Definition 3.3 (carrying capacity)**.** In Verhulst's growth model, the value B/s is called the carrying capacity of the environment, representing the maximum sustainable population under the given conditions.

Note that Verhulst's equation can be easily solved by partial fraction decomposition (see Figure 5 for the graph of the logistic curve). Here, we consider the possibility that we begin with a small population, i.e. $N_0 < B/s$†.

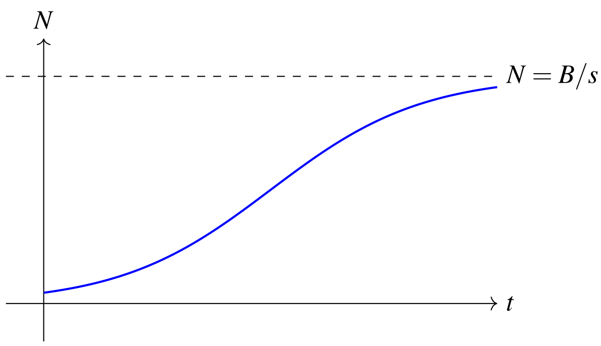

Figure 5: Graph of the logistic function

3.3. *Harvesting*

One major application of mathematical modelling is in dealing with populations of animals. We wish to know how many we can eat (say fish). We will build on Verhulst's model, i.e. assume that the fish population would follow that model if we did not catch any. Next, we assume that we catch H fish per unit time (say year). Then, the new differential equation representing a basic harvesting

†The case where $N_0 > B/s$ will not be discussed. In this other scenario, we assume that we begin with a large population, i.e. $N_0 > B/s$. Then, the solution is monotonically decreasing, but again, the asymptotic value is the same.

model can be written as

$$\frac{dN}{dt} = bN - sN^2 - H.$$

Again, one can use partial fraction decomposition to determine the solution to the differential equation.

4. Systems of First-Order Differential Equations

4.1. *Solving Systems of Ordinary Differential Equations*

Relationships often go through ups and downs. We shall explore a mathematical model to capture this phenomenon. Romeo loves Juliet, but Juliet has a subtler response. When Romeo shows strong affection, Juliet finds his enthusiasm overwhelming, making her feelings for him cool down. However, when Romeo becomes indifferent, Juliet finds him mysteriously attractive. Romeo, on the other hand, reacts more directly: his love for Juliet increases when she is warm and decreases when she is cold.

Let $R(t)$ and $J(t)$ denote Romeo's and Juliet's feelings over time. These feelings can be modelled using the system of first-order linear ordinary differential equations as follows:

$$\frac{dR}{dt} = aJ \text{ and } \frac{dJ}{dt} = -bR \quad \text{where} \quad R(0) = \alpha \text{ and } J(0) = \beta.$$

Here, $a, b > 0$ are positive constants and α, β are initial feelings at $t = 0$. This system describes the interaction between their feelings.

We propose solutions of the form

$$R = Ae^{\lambda t} \quad \text{and} \quad J = Be^{\lambda t}.$$

Note that these can be obtained by transforming the system into two separate second-order linear differential equations, and then construct the characteristic equation to find the solution. Anyway, returning to the Romeo and Juliet problem, substituting R and J into the differential equations yields

$$A\lambda = aB \quad \text{and} \quad B\lambda = -bA.$$

Eliminating A and B, we have $\lambda^2 = -ab$. Since $\lambda^2 < 0$, the solutions are complex, i.e. $\lambda = \pm i\sqrt{ab}$. As such, the general solution can be expressed as a linear combination of sin and cos as follows:

$$R = C\cos\left(\sqrt{ab}t\right) + D\sin\left(\sqrt{ab}t\right) \quad \text{and} \quad J = E\cos\left(\sqrt{ab}t\right) + F\sin\left(\sqrt{ab}t\right)$$

All that is left is to find C, D, E, F. This can be done so by considering the initial conditions. As such,

$$R = \alpha\cos\left(\sqrt{ab}t\right) + \beta\sqrt{\frac{a}{b}}\sin\left(\sqrt{ab}t\right) \quad \text{and} \quad J = \beta\cos\left(\sqrt{ab}t\right) - \alpha\sqrt{\frac{b}{a}}\sin\left(\sqrt{ab}t\right).$$

Motivated by the above, we consider a more general system, i.e.

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

We can write this as a matrix equation, which is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

We consider the solution

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = e^{rt}\mathbf{u}_0 \quad \text{where} \quad \mathbf{u}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

As such,

$$re^{rt}\mathbf{u}_0 = \mathbf{B}e^{rt}\mathbf{u}_0 \quad \text{where} \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

or equivalently,

$$\mathbf{B}\mathbf{u}_0 = r\mathbf{u}_0.$$

This is analogous to the matrix equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, where λ and $\mathbf{v}$ are an eigenvalue and corresponding eigenvector of the matrix $\mathbf{A}$! As such, the possibilities of r are given by the eigenvalues of $\mathbf{B}$. We have

$$(\mathbf{B} - r\mathbf{I})\,\mathbf{u}_0 = \mathbf{0}$$

so non-trivial solutions exist if $\det(\mathbf{B} - r\mathbf{I}) = 0$, i.e. if $(a-r)(d-r) - bc = 0$. Except the case where this quadratic polynomial in r has two repeated roots (i.e. discriminant zero), we must have two solutions r_1 and r_2, which implies

$$\mathbf{u} = c_1e^{r_1t}\mathbf{u}_1 + c_2e^{r_2t}\mathbf{u}_2,$$

where c_1 and c_2 are constants and $\mathbf{u}_1$ and $\mathbf{u}_2$ are the eigenvectors of r_1 and r_2 respectively. Naturally, r_1 and r_2 might be complex so we might have to interpret the exponential functions in terms of sine and cosine.

Example 4.1. Solve

$$\frac{dx}{dt} = -4x + 3y$$
$$\frac{dy}{dt} = -2x + y$$

Solution. In fact, such questions are also covered in MA3220. The matrix representation is

$$\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \quad \text{which} \quad \text{has eigenvalues } -1 \text{ and } -2.$$

The eigenspaces are

$$E_{-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad E_{-2} = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}.$$

The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

In other words,

$$x = c_1e^{-t} + 3c_2e^{-2t} \quad \text{and} \quad y = c_2e^{-t} + 2c_2e^{-2t}.$$

One can deduce the values of c_1 and c_2 given the values of $x(0)$ and $y(0)$. □

5. Modelling with Non-Linear Systems

5.1. *The Lotka-Volterra Model*

Lions like to eat zebras, and depend on them. That is, the lion population goes up if there are many zebras. However, if there were no zebras, then the lions would die out. The zebras eat grass, and would get along just fine if there were no lions. Their population tends to go down when there are lions about but when left to themselves, their population goes up.

Suppose at time t, there are $L(t)$ lions and $Z(t)$ zebras, and assume a Malthusian model for both the lions and zebras in the absence of the other. We shall assume that there is a stab equilibrium at populations (L_0, Z_0). This suggests that we can devise the following model:

$$\frac{dL}{dt} = -(L - L_0) + 2(Z - Z_0)$$
$$\frac{dZ}{dt} = -2(L - L_0) + \frac{1}{2}(Z - Z_0)$$

This resembles the models mentioned in the previous chapter, except that the equilibrium has been shifted from $(0,0)$ to a point in the first quadrant. One can verify that in this case, the equilibrium point is indeed (L_0, Z_0). In fact, it is a type of stable equilibrium — it is a spiral sink. If there is some kind of disturbance, the populations of lions and zebras fluctuate up and down for a while but they eventually get close to equilibrium.

We shall further analyse this model. Suppose $L(0) = 0$ and $Z(0) = 0$. Then, when $t = 0$, $dL/dt = L_0 - 2Z_0$ which is non-zero. The same can be said for dZ/dt when evaluated at $t = 0$. This means that lions and zebras are coming into existence out of nothing or that we will immediately have negative numbers of animals! Moreover, the system we are trying to model has two equilibria — other than (L_0, Z_0), we also have $(0,0)$. That is, it must always be possible to have no lions and no zebras. However, this is not what we will get or with any linear model as such systems only ever have one equilibrium point.

We shall construct a different mathematical model for the lion and zebra situation. This is similar to the logistic model we discussed previously in a sense that the death rate per capita of zebras, D_Z, is not fixed. Here, D_Z depends on the number of lions, so suppose

$$D_Z = sL \quad \text{where} \quad s \text{ is a positive constant.}$$

The constant s tells us something about the relationship between lions and zebras. We continue to assume a Malthusian model for the zebra birth rate per capita, B_Z. As such, we have

$$\frac{dZ}{dt} = B_Z Z - sLZ.$$

What about the lions? When there is a shortage of zebras, they can eat other animals so they will not really starve. However, zebras are *nice and fat*, so the ones which really suffer from a shortage of zebras is not the adult lions but rather, the baby lions. This is because if there is an insufficient number of *nice and fat* zebras around, then the mother lion cannot produce enough milk for the young, and the latter will die. As such, the effect of a shortage of zebras is to reduce the effective birth rate of the lions. Hence, we use a Malthusian model for the death rate of the lions. We write

$$\frac{dL}{dt} = uZL - D_L L.$$

The pair of equations

$$\begin{aligned} \frac{dZ}{dt} &= B_L Z - sLZ \\ \frac{dL}{dt} &= uZL - D_L L \end{aligned}$$

givesa famous model of such populations known as the Lotka-Volterra model, or the predator-prey model. One verifies that $(L,Z) = (B_Z/s, D_L/u)$ is an equilibrium point, and so is $(L,Z) = (0,0)$! As such, we are on the right track. However, the Lotka-Volterra eqautions are non-linear!

5.2. *Linearisation*

Recall the Taylor series expansion for functions of several variables from MA2104/MA3210. Now, suppose that we have a pair of non-linear simultaneous first-order ODEs governing a pair of functions of time $(x(t), y(t))$ of the ofrm

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

and suppose that this system is known to have an equilibrium point at $(x,y) = (a,b)$. This implies $f(a,b) = g(a,b) = 0$. As such, when we obtain the Taylor series expansion for these two functions around the point (a,b), the constant term vanishes! Keeping the linear terms and discarding terms of higher order, we obtain the following equations:

$$\begin{aligned} \frac{dx}{dt} &\approx f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ \frac{dy}{dt} &= g_x(a,b)(x-a) + g_y(a,b)(y-b) \end{aligned}$$

which we can write as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

and now, the equations have become linear! One recalls from MA2104/ST2131 that the matrix of partial derivatives here is known as the Jacobian matrix. In summary, near an equilibrium point, a non-linear system can be approximated by a certain linear system with a matrix given by the Jacobian of the original system. This new system is called the linearisation of the original system. Indeed, this is a good piece of news as we can now apply knowledge from the previous chapter to solve our new pair of differential equations! In particular, the classification theorem of equilibrium points (recall MA3220) now applies. As such, we still have a good idea of what is happening in the phase diagram near to those points — we just have to compute the Jacobian there.

Example 5.1 (lions and zebras)**.** Now, recall the Lotka-Volterra equations for the lion and zebra problem. That is,

$$\frac{dL}{dt} = uZL - D_L L = f(L,Z)$$
$$\frac{dZ}{dt} = B_Z Z - sLZ = g(L,Z)$$

The Jacobian matrix here is

$$\mathbf{J}(L,Z) = \begin{bmatrix} uZ - D_L & uL \\ -sZ & B_Z - sL \end{bmatrix}$$

and we wish to evaluate it at the two equilibrium points.

The first is $(L,Z) = (0,0)$ and so

$$\mathbf{J}(0,0) = \begin{bmatrix} -D_L & 0 \\ 0 & B_Z \end{bmatrix}.$$

It is easy to see that this is a saddle point. Since the matrix $\mathbf{J}(0,0)$ is diagonal, the eigenvalues are the diagonal entries, namely $-D_L$ and B_Z, with corresponding eigenvectors $(1,0)$ and $(0,1)$ respectively. One checks that this makes sense because in the phase plane, everything is rushing towards the origin along the lion axis and away from the origin along the zebra axis. We can expect that if we start nearer to the L-axis, then the lion population will decrease greatly until the zebra populaltion increases rapdily, which makes sense!

The other equilibrium point is $(L,Z) = (B_Z/s, D_L/u)$, and one checks that

$$\mathbf{J}\left(\frac{B_Z}{s}, \frac{D_L}{u}\right) = \begin{bmatrix} 0 & uB_Z/s \\ -sD_L/u & 0 \end{bmatrix}$$

which we recognise as a centre. As such, in the middle of the phase diagram corresponding to large numbers of both lions and zebras, we expect to see the swirling motion. In fact, the direction of motion is clockwise.

Now, take the two Lotka-Volterra equations, multiply the dL/dt equation by $B_Z/L - s$, and multiply the dZ/dt equation by $D_L/Z - u$. Adding the resulting equations, we have

$$\left(\frac{B_Z}{L} - s\right)\frac{dL}{dt} + \left(\frac{D_L}{Z} - u\right)\frac{dZ}{dt} = 0.$$

Hence,

$$B_Z \ln L - sL + D_L \ln Z - uZ = c,$$

where c is an arbitrary constant. This is an exact relation between Z and L although the equation cannot be explicitly solved. One way to draw the graphs is to define a function $F(L,Z)$ on the phase plane by

$$F(L,Z) = B_Z \ln L - sL + D_L \ln Z - uZ.$$

One checks thatt this function has a global minimum at the point $(L,Z) = (B_Z/s, D_L/u)$, with contour curves around that point which are all closed. Since the paths in the phase plane are all closed curves, we see that all solutions of the Lotka-Volterra equations are periodic.

The Lotka-Volterra model can be used to understand an interesting paradox known as the paradox of pesticides. This is the strange observation that when a certain pest has a predator, using pesticides can actually leads to more pests than we had initially!

5.3. *Logistic Lotka-Volterra Model*

The Lotka-Volterra model assumes that the zebra population grows according to a Malthusian model when there are no lions. We know that this is not realistic so we should use something like the logistic model for them, while keeping the old equation for the lions. As such, we obtain the following pair of differential equations:

$$\frac{dL}{dt} = uZL - D_L L$$
$$\frac{dZ}{dt} = B_Z Z - pZ^2 - sLZ$$

Here, p is the logistic constant, so the equilibrium population of zebras would be B_Z/p in the complete absence of lions.

In this model, there are actually three equilibrium points in the phase diagram. The first is the obvious one $(0,0)$. The second is almost as obvious, i.e. if there are no lions, then the zebras will approach a logistic equilibrium along the Z-axis, i.e. the point $(0, B_Z/p)$. The third and most interesting one is at

$$\left(\frac{B_Z - pD_L/u}{s}, \frac{D_L}{u}\right).$$

We omit the remaining details.

6. Modelling with Partial Differential Equations

6.1. *Introduction*

Definition 6.1 (partial differential equation)**.** A partial differential equation (PDE) is an equation containing an unknown function $u(x,y,\ldots)$ of two or more independent variables $x,y,\ldots$ and its partial derivatives with respect to these variables. We call u the dependent variable.

PDEs allow us to deal with situations where something depends on space as well as time. So far, all the models that we studied so far have only involved variations with time.

We discuss a method to solve PDEs known as the separation of variables. This method can be used to solve PDEs involving two independent variables say x and y that can be separated from each other in the PDE. There are similarities between this method and the technique of separating variables for ODEs in the first chapter.

We make the following observation. Suppose

$$u(x,y) = X(x)Y(y).$$

Then,

$$\begin{aligned} u_x &= X'(x)Y(y) \\ u_y &= X(x)Y'(y) \\ u_{xx} &= X''(x)Y(y) \\ u_{yy} &= X(x)Y''(y) \\ u_{xy} &= X'(x)Y'(y) \end{aligned}$$

Note that each derivative of u remains separated as a product of a function of x and a function of y. We can exploit this feature. Consider a PDE of the form

$$u_x = f(x)g(y)u_y.$$

If a solution of the form $u(x,y) = X(x)Y(y)$ exists, then one can deduce that

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = g(y) \cdot \frac{Y'(y)}{Y(y)}.$$

The important observation here is that the LHS is a function of x whereas the RHS is a function of y. We conclude that the LHS and RHS both equate to some constant k. As such, we obtain the two ODEs as follows:

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = k \quad \text{and} \quad g(y) \cdot \frac{Y'(y)}{Y(y)} = k.$$

In fact, it is easy to solve this pair of differential equations!

Example 6.1. Solve $u_x + xu_y = 0$.

Solution. Suppose a solution of the form $u(x,y) = X(x)Y(y)$ exists. Then, we deduce that

$$X'(x)Y(y) + xX(x)Y'(y) = 0$$
$$\frac{1}{x} \cdot \frac{X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)}$$

As such,

$$\frac{1}{x} \cdot \frac{X'(x)}{X(x)} = k \quad \text{and} \quad -\frac{Y'(y)}{Y(y)} = k.$$

This implies that $X(x) = ae^{kx^2/2}$ and $Y = be^{-ky}$ for some constants aA and b. As such, the general solution is

$$u(x,y) = X(x)Y(y) = ce^{kx^2/2-ky}.$$

Here, $c = ab$ is also a constant. □

6.2. *The Wave Equation*

Consider a flexible string that lies stretched tightly (another word would be 'taut') along the x-axis and has its ends fixed at $x = 0$ and $x = \pi$. We pull it along the u-axis so that it is stationary and has some specific shape $u = f(x)$ at time $t = 0$. Consequently, $f(0) = 0$ and $f(\pi) = 0$. We can assume that $f(x)$ is continuous and bounded. When we let go of the string, it will move. We assume that the only forces acting are those due to the tension in the string and that the pieces of the string will only move along the u-axis.

Now, the u-coordinate of any point on the string will become a function of time as well as a function of x. So, it becomes a function $u(t,x)$ of both t and x. Note that this function satisfies the boundary conditions

$$u(t,0) = 0 \quad \text{and} \quad u(t,\pi) = 0$$

for all t as the ends are nailed down. Also, the initial condition

$$u(0,x) = f(x) \quad \text{is satisfied.}$$

Also, since the string is initially stationary, then

$$\frac{\partial u}{\partial t}(0,x) = 0.$$

We now introduce the wave equation.

Definition 6.2 (wave equation)**.** Let c be a fixed non-negative real constant representing the propagation speed of the wave. Then,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Also, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = f(x)$ and u_t evaluated at $t = 0$ gives 0.

We note that the function

$$u(t,x) = \frac{f(x+ct) + f(x-ct)}{2} \quad \text{is a solution to the wave equation.}$$

More generally, we have D'Alembert's formula (Theorem 6.1). One should check that the above equation indeed satisfies the wave equation. Moreover, the four conditions should be satisfied.

Theorem 6.1 (D'Alembert's solution to the wave equation)**.** The function

$$u(t,x) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

is a solution to the wave equation (Definition 6.2).

Initially, $f(x)$ was only defined between $x = 0$ and $x = \pi$, but the interpretation of D'Alembert's solution is that we can extend $f(x)$ to be an odd, periodic function of period 2π, for which one can then verify that $u(t,0) = u(t,\pi) = 0$.

Consider a function $f(x)$ defined on $[0,\pi]$. We aim to extend f to an odd periodic function with periodic 2π, covering $\mathbb{R}$. First, we define f on $[-\pi,\pi]$ as an odd function, i.e. for any $x \in [-\pi,0]$, define $f(-x) = -f(x)$. This extension makes f an odd function on $[-\pi,\pi]$. Next, we extend f periodically across the real line, i.e. for any x outside $[-\pi,\pi]$, define

$$f(x) = f(x - 2n\pi) \quad \text{where } n \in \mathbb{Z} \text{ and } x - 2n\pi \in [-\pi,\pi].$$

This creates a periodic function with period 2π.

We then consider the function $f(x-ct)$, where $c > 0$. Note that $f(x-1)$ is the same as $f(x)$, just shifted 1 unit to the right. Similarly, $f(x-ct)$ represents the same shape as $f(x)$ but shifted to the right by ct units. The function $f(x+ct)$ represents $f(x)$ shifted to the left by ct, moving in the opposite direction with the same speed.

Geometrically, D'Alembert's solution describes the solution to the wave equation as a combination of two travelling waves, where the term

$$\begin{aligned} &f(x-ct)/2 \quad \text{representsa wave traveling to the right at speed } c \quad \text{and} \\ &f(x+ct)/2 \quad \text{representsa wave traveling to the left at speed } c \end{aligned}$$

Each piece maintains the shape of the original function $f(x)/2$ and moves without distortion.

We can also solve the wave equation using the method f separation of variables. Suppose we wish to solve

$$u_{tt} = c^2 u_{xx}$$

subjected to the following conditions:

$$u(t,0) = u(t,\pi) = 0 \quad u(0,x) = f(x) \quad u_t(0,x) = 0$$

We separate the variables, i.e.

$$u(t,x) = v(x)w(t)$$

and obtain

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = -\lambda.$$

The usual separation argument from before implies λ is a constant, so we obtain the following pair of ODEs:

$$v'' + \lambda v = 0 \quad \text{and} \quad w'' + \lambda c^2 w = 0$$

Let us force $v(x)$ to vanish at $x = 0$ and $x = \pi$, so we can set

$$u(t,0) = v(0)w(t) = 0 \quad \text{and} \quad u(t,\pi) = v(\pi)w(t) = 0.$$

This is somewhat different from the usual scenario of solving second-order ODEs. Normally, we give some information about the function at one point, i.e. we might ask for solutions to the ODE $y'' + \lambda y = 0$ where $y(0)$ and $y'(0)$ are given. However, we are now giving the information from two different points.

If $\lambda < 0$, then $u(0) = 0$ implies that all solutions to the equation $v'' + \lambda v = 0$ are proportional to $\sinh x$, and such a function cannot intersect the x-axis twice. As such, λ cannot be negative. If $\lambda = 0$, then $v(x)$ is a straight line function which cannot intersect the x-axis twice. As such, $\lambda > 0$. We write $\lambda = n^2$ for some $n > 0$. As such,

$$v(x) = C\cos nx + D\sin x \quad \text{for some constants } C \text{ and } D.$$

Since $v(0) = 0$, then $C = 0$ and so $v(x) = D\sin nx$. If we want $v(\pi) = 0$, then it implies $\sin n\pi = 0$. As such, $n \in \mathbb{Z}$ so we also introduce this constraint. Earlier, we mentioned that $n > 0$. Combining both properties, we conclude that $n \in \mathbb{Z}^+$.

Solving the other equation for $w(t)$, we obtain

$$w(t) = A\cos nct + B\sin nct \quad \text{for some constants } A \text{ and } B.$$

We force $w(t)$ to satisfy $w'(0) = 0$ since we want $u_t(0,x) = u(x)v'(0) = 0$. So, now $B = 0$ and we are left with $w(t) = A\cos nct$. As such, our complete solution is

$$u(t,x) = b_n \sin nx \cos nct.$$

Here, b_n is an arbitrary constant and again, recall that n is a positive integer. This satisfies three of the four conditions in Definition 6.2. So, the only condition that is not yet satisfied is $u(0,x) = f(x)$.

We recall some concepts from MA2101. Think about the set of all continuous functions on $[0,\pi]$. It is a vector space over $\mathbb{R}$ (an obvious fact). What is a possible basis for it? Well, an example of a basis is given by the following set:

$$\left\{\sin nx : n \in \mathbb{Z}^+\right\}$$

In other words, any continuous function g on $[0,\pi]$ can be expressed as the following:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

This holds for certain real numbers b_n. In particular, the formula for b_n is

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\,dx.$$

That is, $2/\pi$ times the integral plays the role of the scalar product here. The series

$$\sum_{n=1}^{\infty} b_n \sin x \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\,dx$$

is known as the Fourier series of $g(x)$. There is an amazing fact that the Fourier series allows us to express any function on this interval as the components — b_n! We now return to the problem of solving the wave equation. Recall that we have extended $f(x)$ to be an odd function of period 2π. As such, it has a Fourier sine series, and since f is continuous and has only a finite number of sharp corners, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Now, consider the series

$$\sum_{n=1}^{\infty} b_n \sin nx \cos nct \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

First, observe that if we substitute $t = 0$ in this series, then we obtain $f(x)$ expressed as its Fourier sine series. Next, since the wave equation is linear and each term in this series is a solution to the wave equation, then this series is also a solution to the wave equation.

To summarise, the solution to the wave equation is

$$u(t,x) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \right) \sin nx \cos nct$$

We have done everything for the interval $[0,\pi]$. For a general interval $[0,L]$ of any length L, it is easy to obtain a solution to the modified wave equation. The basis functions are now $\sin(n\pi x/L)$ which are periodic with period $2L$ instead of 2π like before. The Fourier series formulae are

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad b_n = \frac{2}{L} \int_0^{L} g(x) \sin\left(\frac{n\pi x}{L}\right) \, dx.$$

f will now be a function that vanishes at 0 and L!

6.3. *The Heat Equation*

Consider the temperature in a long thin bar or wire of constant cross-section and homogeneous material which is oriented along the x-axis and is perfectly insulated laterally, so that heat only flows in the x-direction. Then the temperature u depends only on x and t and is given by the one-dimensional heat equation.

> **Definition 6.3** (heat equation). The heat equation states that
>
> $$u_t = c^2 u_{xx},$$
>
> where c^2 is a positive constant called the thermal diffusivity (sometimes this is denoted by κ). It measures how quickly heat moves through the bar and depends on what it is made of.

Let us assume that the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the following boundary conditions:

$$u(0,t) = 0 \quad u(L,t) = 0 \quad \text{for all } t,$$

and the initial temperature of the bar is $f(x)$, so that we have the initial condition

$$u(x,0) = f(x).$$

Here, we will assume that when $f(x)$ is extended to be an odd function, it equals its Fourier sine series everywhere. Remember that this can happen, even if $f(x)$ is discontinuous at some points.

Notice that, unlike the wave equation, which needs four pieces of data, here we only need three, which matches the fact that the heat equation only involves a total of three derivatives (two in the spatial direction, but only one in the time direction).

The heat equation is particularly useful in modeling for the following reason. Think of an ordinary function, $g(x)$. We can think of its second derivative $g''(x)$ as a measure of the extent to which its graph is not a straight line (recall that the second derivative is zero everywhere if and only if $g(x)$ is a linear function). We say that $g''(x)$ measures the curvature of the graph.

The heat equation says that the second spatial derivative of u is equal to its time derivative. So as time goes by, if the graph of u as a function of x is concave up, then u will increase; whereas if the graph is concave down, then it tends to decrease. The effect in both cases is to reduce the curvature. So we can picture the equation as something that, given an initial shape described by $f(x)$, tries to "straighten it out." And of course, that is how we expect heat to behave, i.e. heat flows from a hotter region to a cooler region, trying to even out its distribution.

It turns out that the solution of the one-dimensional heat equation looks like this.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2c^2}{L^2}t\right),$$

where the b_n are just the Fourier sine coefficients of $f(x)$.

Notice that we get exponentials here instead of sines. That is because the separated equation for the function of t is first-order (for obvious reasons), and as we know, first-order ordinary differential equations tend to have exponential solutions. Because of this, the solutions to the heat equation depend on the direction of time. This means that this PDE is useful for modeling situations involving irreversible time evolution.

6.4. *Fisher's Equation*

Life on dry land took a long time to evolve: animals and plants had lived in the sea for hundreds of millions of years before that happened, roughly 450 million years ago. Of course, it must have started along the sea shore, that is, along a line. There must have been some kind of marine plant growing along the shore line; a mutation occurred (helped by the extreme exposure to sunlight) which made one of them, at some particular time and place, better able to tolerate drying out. The descendants of that individual had a tremendous advantage over the non-mutated neighbours because sometimes there is a succession of exceptionally low tides which leave the plants dry for a long time. So they would have outcompeted their neighbours, and the mutation would have spread along the shoreline like a wave. Eventually, the result would be a plant that could survive out of the water full-time.

The process of spreading along the shoreline is clearly irreversible, so we need an equation like the heat equation, not the wave equation: we need a heat equation with a wave-like solution! On the other hand, we do not want the effect to go away, like the temperature going down as heat dissipates. What we need is a combination of the Heat Equation with our model of the spread of a rumour. In 1937, Ronald Fisher proposed the following equation to model this situation:

$$u_t = \alpha u_{xx} + \beta u(1-u)$$

where $u(x,t)$ is the fraction of the plants at any given place and time which have mutated (so $1-u(x,t)$ is the fraction which haven't). This is indeed a combination of the Heat Equation with the rumour equation! The constant α tells you how quickly the mutation tends to spread in space, while β measures how quickly it grows in time at a specific point in space (they have different units, of course).

This is a non-linear partial differential equation, and finding all of its solutions is very difficult. But it is important because it has many other applications, for example to the theory of how flames move and to the theory of how nuclear reactors work. To solve this equation, we specify some initial function $f(x) = u(x,0)$ and then try to evolve it forward in time. A good model for $f(x)$ would be a delta function.

We seek a wave solution of the form

$$u(x,t) = U(x-ct)$$

where $U(s)$, $s = x - ct$, describes the wave moving to the right at constant speed c. Substituting this into Fisher's equation gives

$$\alpha U'' + cU' + \beta U - \beta U^2 = 0.$$

This ODE has two equilibria: $(U,U') = (0,0)$ and $(U,U') = (1,0)$. The Jacobian at these equilibria determines their stability. We have

$$\mathbf{J}(U,U') = \begin{bmatrix} 0 & 1 \\ \frac{\beta}{\alpha}(2U-1) & -\frac{c}{\alpha} \end{bmatrix}.$$

We omit the remaining details.